Solution of 2-D Acoustic Wave Equation with Numeric Methods

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ABSTRACT

.

In this study, two-dimensional acoustic wave equation was solved with the finite differences, Fourier and variable scale methods. Two-dimensional wave propagation modeling was performed and boundary conditions were researched for the solution. The methods used in the modeling, source functions and differences were explained, various seismograms were formed and the used methods were compared. Only the window boundaries are used for the modeling in the Fourier method. Therefore, some phenomena are absorbed in the first and last traces in seismograms. Phenomena are observed in all the traces when the finite difference and variable scale methods are used. Thus, one of these two methods can be preferred for modeling complex structures. Variable-scale method is more useful than the finite difference and Fourier methods for one borehole geometry.

Key Words: 2-D Acoustic wave, Finite differences method, Fourier method, Variable scale method, Modeling, Boundary conditions, Complex structures, Seismogram

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1. INTRODUCTION

Generating synthetic seismograms and comparing them with real seismograms provide quite useful information in the applied geophysics. Studies have been conducted on this topic widely and successfully for many years. The increasing interest in seismic modeling led to the development of the methods with diverse accuracies and providing application convenience. Together with the increase in computer capabilities in 1970s, wave equations were solved with various numeric methods and studies started on the synthetic seismogram modeling.

Finite differences approximation [1-5] was used successfully to solve the wave propagation problems. Later, the finite elements method [6] began to be used. Modeling studies started with the Fourier transform as an alternative to both of these methods in 1980s [7-10]. In following years, the Hartley transform, which had a quicker algorithm, was given as an alternative to the modeling with the Fourier transform in 1990 and 1991 [11-12]. The finite differences method is used for solving the drilling activities. Solving them is quite difficult with this method because the drilling hole is very small. Therefore, the variable scale method (VSM) [13], which was developed to calculate a small part of the deeper and wider drilling grate, is very useful at Cartesian and cylindrical coordinates. Irregular grids were used firstly by Jensen [14]. Afterwards, it was applied to the finite differences by Perrone, Kao [15], Demir [16] and Benito [17], respectively.

Complicated structures in the ground are researched by means of synthetic seismograms. The contribution of these data is great for learning how the micro or global changes in the ground influence the wave forms and for interpreting the seismograms in the seismic stratigraphy. The interior of the ground has a complicated structure although it is assumed to consist of homogeneous layers many times. Diverse methods are used for synthetic seismogram modeling.

The elastic wave equation includes P and S waves because it is comprised of horizontal and vertical components; hence, the wave reaching the boundary that separates the two media with different characteristics is reflected, broken and goes through mode transformation. And the acoustic wave is formed of only the P waves, because it causes longitudinal vibration. Apart from the specific purposes, only the P waves are recorded in the seismic prospecting and the S waves, which can be obtained in these records, are defined as noise. Therefore, the acoustic wave equation is used in modeling studies for seismic prospecting.

The acoustic wave equation can be solved with various numeric methods. In this study, the two-dimensional acoustic wave equation was solved with the finite differences, Fourier and variable scale (VSM) methods, and various modeling's were implemented.

2. 2-D Acoustic Wave Equation

The source function is defined as $f(x,z,t)$ for the two-dimensional medium. Velocity is considered as $c(x,z)$ for the two-dimensional medium and the two-dimensional acoustic wave equation [18] is stated as follows:

$$\frac{1}{c^2(x,z)}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+f(x,z,t) \qquad (1)$$

2.1. Finite Differences Approximation

The finite differences approximation is a method especially and extensively used in solving the partial differential equations. This method is applied in three ways;
namely the forward-differences, backward differences and centre-differences. [19]. Now, the value of a function, which has the $u(x)$ value at X point, will be considered as $u(x+\Delta x)$ at the $x+\Delta x$ point and $u(x-\Delta x)$ at the $x-\Delta x$ point. For such a function, its derivative at x point is stated in terms of finite differences as follows:
The first $\partial U/\partial x$ derivative can be written in terms of forward-differences as

$$\frac{\partial u}{\partial x}=\frac{1}{\Delta x}\{u(x+\Delta x)-u(x)\} \tag{2}$$

In terms of backward-differences as

$$\frac{\partial u}{\partial x}=\frac{1}{\Delta x}\{u(x)-u(x-\Delta x)\} \tag{3}$$

In terms of centre-differences as

$$\frac{\partial u}{\partial x}=\frac{1}{2\Delta x}\{u(x+\Delta x)-u(x-\Delta x)\} \tag{4}$$

And the $\partial^2 u/\partial x^2$ derivative can be written by means of the central-finite differences as [19]

$$\frac{\partial^2 u}{\partial x^2}=\frac{u(x+\Delta x)-2u(x)+u(x-\Delta x)}{\Delta x^2} \tag{5}$$

2.2. Fourier Approximation

The Fourier transform pair for a $f(t)$ function observed in the time medium is as follows [20]:

$$\begin{aligned} F(w) &= \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}\,dw \end{aligned} \tag{6}$$

Here, w is the angular frequency and $w=2nf(t)$. The equivalents of discrete and finite data in the frequency and time media are defined as [20].

$$\begin{aligned} F(w) &= \sum_{t=0}^{N-1} f(t)[\cos(wt/N)-i\sin(wt/N)] \\ f(t) &= \sum_{w=0}^{N-1} F(w)[\cos(wt/N)+i\sin(wt/N)] \end{aligned} \tag{7}$$

In the formula, N is the number of data; the Nyquist frequency is equal to $f(t)N=1/2\Delta t$ and showed with the $\Delta f(t)=1/N\Delta t$ value. There was a changeover from the distance medium to the wave number medium in this study. In these two media, the discrete Fourier transform pair is stated as

$$\begin{aligned} F(k_x) &= \sum_{x=0}^{N-1} f(x)[\cos(k_x x/N)-i\sin(k_x x/N)] \\ f(x) &= \sum_{k_x=0}^{N-1} F(k_x)[\cos(k_x x/N)+i\sin(k_x x/N)] \end{aligned} \tag{8}$$

Here, k_x is the number of waves and has the value of

$$k_x = \frac{2\pi n}{N\Delta x} \quad (9)$$

x corresponds to the spatial sampling interval and n=1,2,3,……,N.

The second-order $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial z^2$ derivatives in the acoustic wave equation given with the equation (1) in the modeling study will be calculated through the Fourier transform. The first and second derivatives of an f function observed in the distance medium with the help of the derivative attributes of the Fourier transform are written in terms of x as follows [20].

$$\frac{\partial f}{\partial x} \xleftrightarrow{F} ik_x F(k_x) \quad (10)$$

$$\frac{\partial^2 f}{\partial x^2} \xleftrightarrow{F} -k_{x^2} F(k_x) \quad (11)$$

For the Fourier transform, the amplitude spectrum is defined as follows:

$$|F(w)| = \left(\operatorname{Re} F(iw)^2 + \operatorname{Im} F(iw)^2\right)^{1/2} \quad (12)$$

2.3. Variable Scale Method

This method is quite useful for the Cartesian and cylindrical coordinates [13].

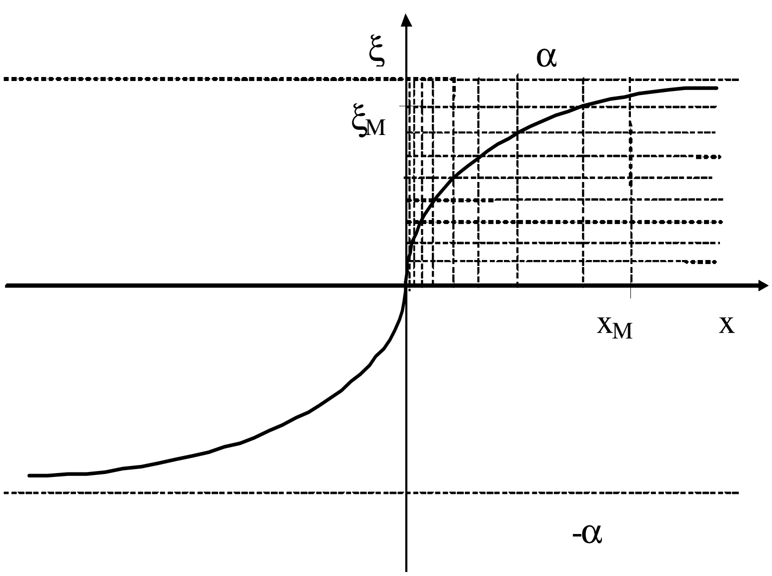

Figure 1.1 Variable scale geometry

The following equation has been obtained with the variable scale method:

$$\xi = \alpha \tanh(\beta x) \quad (13)$$

α, β is a positive constant.

The 2-D wave equation has been converted for the variable scale method as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial \xi}\left(A(\xi)\frac{\partial u}{\partial \xi}\right) + c^2 \frac{\partial^2 u}{\partial z^2} \tag{14}$$

Here,

$$A(\xi) = dx/d\xi = (\beta/\alpha)(\alpha^2-\xi^2)$$

is considered.

3. Solving the 2-D Acoustic Wave Equation with Numeric Methods

When an analytical solution cannot be obtained for solving and modeling the wave equation of a seismogram belonging to the complicated underground, the numeric analysis methods are used directly. Very fast and high-capacity computers are needed to solve the acoustic wave equation with the finite differences and the Fourier transform. The developments in the computer technology provide this opportunity.

In the study, the two-dimensional acoustic wave equation will be solved numerically through the finite differences and the Fourier transform.

3.1. Solving the 2-D Acoustic Wave Equation with the Finite Differences

The derivative equalities in the 2-D acoustic wave equation can be written as follows

$$\frac{\partial^2 u}{\partial x^2} = \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} \tag{15}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{\Delta z^2} \tag{16}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} \tag{17}$$

by using the forward-finite differences [18]. Δt defines the time sampling interval; Δx and Δz are the sampling intervals in x and z directions respectively. $\Delta x=\Delta z=h$ is considered so that the operations, which will be carried out, can be easier and quicker. i,j and n are the indexes corresponding to the x (expansion direction), z (depth) and t (time) parameters respectively. If the equations (15), (16) and (17) stated above are substituted in the wave equation given in (1),

$$\frac{1}{c^2(x,z)}\frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} = \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{h^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2} + f(x,z,t) \tag{18}$$

is obtained. When this equation is reformulated by considering $P=c\Delta t/h$, the equation

$$u^{n+1}_{i,j} = 2(1-2P^2)u^n_{i,j} - u^{n-1}_{i,j} + P^2\left[u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j+1} + u^n_{i,j-1}\right] + f(x,z,t) \tag{19}$$

is obtained [5]. The grid used for solving the equation is given below (Figure 2.1).

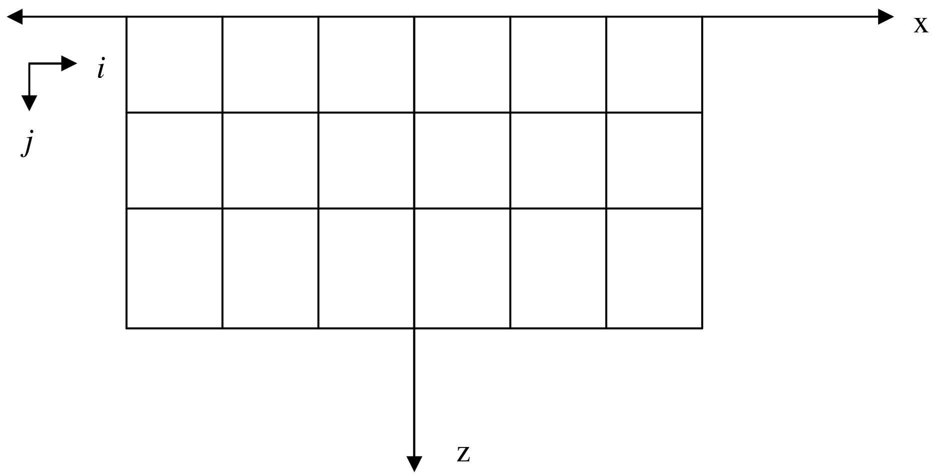

Figure 2.1. 2-D grid network is used to solve the acoustic wave equation

The equation (19) was solved between -$a \le x \le a$, $0 \le z \le b$ boundaries, physical explanations of which are given below for the 2-D medium (Figure 2.2).

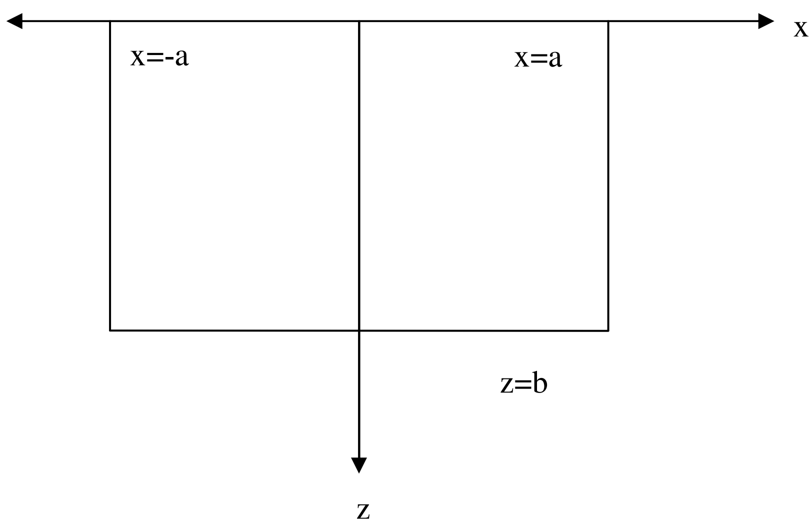

Figure 2.2. 2-D boundary used in the solution of the acoustic wave equation

For the solution,

$$u(x,z,0)=0$$
$$\frac{\partial u(x,z,0)}{\partial t}=0 \qquad (20)$$

was accepted. The values given in the equation (20) are stated as

$$u_{i,j}^{0}=u_{i,j}^{1}=0$$
$$1 \le i \le I+1,\ 1 \le j \le J+1 \qquad (21)$$

in terms of the finite differences [5].

3.2. Solving the 2-D Acoustic Wave Equation with the Fourier Approximation

All the derivative equalities are calculated with the forward-finite differences while the 2-D acoustic wave equation is solved with the finite differences and modeling is conducted. During the modeling with the Fourier method, the spatial $\left(\partial^2 u/\partial x^2 \;\; and \;\; \partial^2 u/\partial z^2\right)$ derivatives are calculated with the Fourier transform and the temporal derivative with the $\partial^2 u/\partial t^2$ finite differences. The $\partial^2 u/\partial t^2$ derivative is given with the finite differences and the equation (17). Other derivative equalities are stated as

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 F(k_x) \qquad (22)$$

Here, k_x, k_z point at the number of waves respectively for the x and z directions. The derivative operators on the left side of the equation (1) is stated for the Fourier transform as

$\left(\partial^2 u/\partial x^2 + \partial^2 u/\partial z^2\right)$,

$$DERIVATIVE\_u(i, j) = -k_{x^2} F(k_x) - k_{z^2} F(k_z) \qquad (23)$$

If the equation (23) is substituted in the acoustic wave equation given with (1) during the modeling with the Fourier transform,

$$\frac{1}{c^2(x,z)} \frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} = DERIVATIVE\_u(i, j) + f(x,z,t) \qquad (24)$$

is obtained. If this last statement is reformulated, the equation

$$u_{i,j}^{n+1},_{k+1} = c^2(x,z)\Delta t^2 \left(DERIVATIVE\_u(i, j) + f(x,z,t)\right) + 2u_{i,j}^{n} - u_{i,j}^{n-1} \qquad (25\ )$$

is obtained [11]. The grid (Figure 2.1) given in the section of solving with the finite differences was used while the equation (25) was solved with the Fourier transform. Besides, the initial condition given with the equation (21) was also used for this method. x series were formed by selecting the grid points parallel to the x-axis while calculating the $\partial^2 u/\partial x^2$ derivative. After the values found with the transformation were multiplied by $-k_x^2$ and the inverse transformation was calculated, the spatial medium was considered. The $\partial^2 u/\partial z^2$ derivative was calculated similarly. Grids parallel to the z-axis were selected and the values found with the transformation were multiplied by $-k_z^2$, then the inverse transformation was calculated and the spatial medium was considered again [18]. Afterwards, these two derivative values were added up and the DERIVATIVE_$u(i,j)$ values were determined (For the Fourier transform (23). The $\partial^2 u/\partial t^2$ derivative was calculated as explained with the equation (17) in the section of modeling with the finite differences, and the values on the grid were determined for the first time step when these values were substituted in the equation (25). Modeling was completed after all of these operations were repeated for all the time steps.

3.3. Solving the 2-D Acoustic Wave Equation with the Variable Scale Method (VSM)

The 2-D acoustic wave equation (1) was solved with the variable scale method and the finite differences (16), then

$$\begin{aligned} u_{i,j}^{n+1} &= 2u_{i,j}^{n} - u_{i,j}^{n-1} + P^2 A_i \left(A_{i+1} u_{i+1,j}^{n} - (A_{i+1} + A_i)) u_{i,j}^{n} + A_i u_{i-1,j}^{n}\right) \\ &+ 2P^2 B_j \left[B_{j+1} u_{i,j+1}^{n} - (B_{j+1} + B_j) u_{i,j}^{n} + B_j u_{i,j-1}^{n}\right] + P^2 f(t) \end{aligned} \quad (26)$$

was obtained.

4. Boundary conditions for solving the Acoustic Wave Equation

One of the difficulties experienced in the synthetic seismogram modeling for solving the wave equation numerically is the selection of the appropriate boundary conditions. The underground model must be restricted in horizontal and vertical directions in the seismogram calculations. If appropriate boundary conditions are not used, synthetic discontinuities will emerge in horizontal and vertical directions. These synthetic discontinuities are called “boundary (edge) reflections” [3]. Different methods were suggested to suppress the boundary reflections. These methods are applied generally in two ways. In the first method, the wave equation is divided into wave fields heading right, left and down, and the boundary levels are determined from the plane waves heading towards these boundaries [5, 20]. This method is applied to the solutions conducted with the finite differences. The second method is applied to the modeling’s performed with transformation and includes windowing the grid edges with the appropriate window [21]. It is difficult to use the boundary conditions for the Fourier transform, which are applied through the division of the wave fields; because the transformations will take place in the lines or the columns. Within the finite differences, the boundary values are determined beforehand, and then the inner points are defined. The windowing method can also be used for the modeling with the finite differences. The first method will be applied to the finite differences and the second method to the Fourier transform in order to explain these boundaries used in the implementation, and it will be given how these methods were applied.

4.1 Boundary Conditions for the Finite Difference Method.

The 2-D acoustic wave equation will be solved for $-a \le x \le a$, $0 \le z \le b$ and $t \ge 0$. These physical boundaries are stated in Figure 2.2. Appropriate boundary conditions must be chosen so that the wave does not go through reflection on the edges at $x=\pm a$ and $z=b$. These boundary conditions can be considered as

$$\begin{aligned} u(\pm a, z, t) &= 0 \\ u(x, b, t) &= 0 \end{aligned} \quad (27)$$

or

$$\begin{aligned} \frac{\partial u}{\partial x}(\pm a, z, t) &= 0 \\ \frac{\partial u}{\partial z}(x, b, t) &= 0 \end{aligned} \quad (28)$$

and the solution of this wave equation is

$$u(x,z,t) = e^{i(wt - kx\cos\theta \pm kz\sin\theta)} + \mathrm{R}e^{i(wt + kx\cos\theta \pm kz\sin\theta)} \quad (29)$$

Here, θ is the angle between the x-axis and the wave front, and R is the reflection coefficient. If this equality is solved for R with its substitution in (27) or (28), the reflection coefficient will be 1. This shows that the coming wave has a complete back-reflection from the boundary. Appropriate boundary conditions must be chosen to suppress these reflections. If the wave field is divided as the fields heading right, left and down from the source, a plane wave field heading left is defined as [5].

$$\left[\frac{1}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right]\left[\frac{P}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right]u=0 \qquad (30)$$
$$x=-a,\quad 0\le z\le b \quad,\quad 0<t\le T$$

and a plane wave field heading right as

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right]\left[\frac{P}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right]u=0 \qquad (31)$$
$$x=a,\quad 0\le z\le b \quad,\quad 0<t\le T$$

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right]\left[\frac{P}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right]u=0 \qquad (32)$$
$$-a\le x\le a,\quad z=b \quad,\quad 0<t\le T$$

Here, $P=c\Delta t/h$. If the $u(x,z,t)$ solution given in (29) is substituted in (30), (31) or (32) and solved for R, the reflection coefficient will be zero. This shows the accuracy of the boundary conditions. The equations (30), (31), (32) can be written in terms of the finite differences respectively as follows [5].

$$u_{1,j}^{n+1}=u_{1,j}^{n}+u_{2,j}^{n}-u_{2,j}^{n-1}+P\left[u_{2,j}^{n}-u_{1,j}^{n}-(u_{3,j}^{n-1}-u_{2,j}^{n-1})\right] \qquad (33)$$
$$2\le j\le J \quad,\quad 2\le n\le N$$

$$u_{I+1,j}^{n+1}=u_{I+1,j}^{n}+u_{I,j}^{n}-u_{I,j}^{n-1}+P\left[u_{I+1,j}^{n}-u_{I,j}^{n}-(u_{I,j}^{n-1}-u_{I-1,j}^{n-1})\right] \qquad (34)$$
$$2\le j\le J \quad,\quad 2\le n\le N$$

$$u_{i,J+1}^{n+1}=u_{i,J+1}^{n}+u_{i,J}^{n}-u_{i,J}^{n-1}+P\left[u_{i,J+1}^{n}-u_{i,J}^{n}-(u_{i,J}^{n-1}-uu_{i,J-1}^{n-1})\right] \qquad (35)$$
$$2\le i\le I \quad,\quad 2\le n\le N$$

4.2 Boundary Conditions for the Fourier Transform

Data windowing is commonly used to suppress the edge reflections during the modeling with the Fourier transform. One of the window functions such as Hanning, Hamming, Blackman, triangle etc. can be used as the window function. In this study, Hanning window was used. Hanning window is defined with the equation

$$W(t)=\begin{cases}0.5+0.5\cos\left(\dfrac{2\pi t}{T}\right) & ,\quad |t|\le T/2 \\ 0 & ,\quad |t|>T/2\end{cases}, \qquad (36)$$

in the time medium. Here, t points at the time and T at the period. In Hanning window with a 1s sampling interval. Windowing corresponds to multiplication in the time medium and convolution in the frequency medium. The values determined with the equation (25) at each time step are multiplied by the window function to suppress the reflections from the left, right and lower sides of the 2-D model.

The values of the model in *x* and *z* directions are windowed to suppress the edge reflections in the 2-D modeling.

4.3. Boundary Conditions for the Variable Scale Method

Absorbing boundary conditions for the variable scale method

For $x = \pm$ a:

$$\frac{P}{c^2}\frac{\partial^2 u}{\partial t^2} \pm A(\mp a)\left(\frac{P+1}{c}\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{37}$$

and for $z = \pm$ a:

$$\frac{P}{c^2}\frac{\partial^2 u}{\partial t^2} \pm B(\mp a)\left(\frac{P+1}{c}\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial^2 u}{\partial z^2}\right) = 0 \tag{38}$$

Absorbing boundary conditions for the VSM, the equations (19) and (20) were modified by using the finite differences.

$$\begin{aligned}
&u_{0,j}^{n+1} = u_{0,j}^{n} + u_{1,j}^{n} - u_{1,j}^{n-1} + A(0)P\left(u_{1,j}^{n} - u_{0,j}^{n} - \left(u_{2,j}^{n-1} - u_{1,j}^{n-1}\right)\right)\\
&\quad 1 \le j \le N,\ 1 \le n \le J,\ A(0) = 0.15963\\
&u_{M,j}^{n+1} = u_{M,j}^{n} + u_{M-1,j}^{n} - u_{M-1,j}^{n-1} - A(M)P\left(u_{M,j}^{n} - u_{M-1,j}^{n} - \left(u_{M-1,j}^{n-1} - u_{M-2,j}^{n-1}\right)\right)\\
&\quad 1 \le j \le N,\ 1 \le n \le J,\ A(N) = 0.14966\\
&u_{i,0}^{n+1} = u_{i,0}^{n} + u_{i,1}^{n} - u_{i,1}^{n-1} + B(0)P\left(u_{i,1}^{n} - u_{i,0}^{n} - \left(u_{i,2}^{n-1} - u_{i,1}^{n-1}\right)\right)\\
&\quad 1 \le i \le M,\ 1 \le n \le J,\ B(0) = 0.15963\\
&u_{i,N}^{n+1} = u_{i,N}^{n} + u_{i,N-1}^{n} - u_{i,N-1}^{n-1} - B(N)P\left(u_{i,Nj}^{n} - u_{i,N-1}^{n} - \left(u_{i,N-1}^{n-1} - u_{i,N-2}^{n-1}\right)\right)\\
&\quad 1 \le i \le M,\ 1 \le n \le J,\ B(N) = 0.14966
\end{aligned} \tag{39}$$

5. Stability Condition and Grid Dispersion

Some conditions must be fulfilled for solving the acoustic wave equation numerically. Selection of the plane sampling intervals (Δx, Δz) and the time sampling interval depends on some criteria. When these criteria are exceeded, values will be found, which are more different than what is supposed to be found, and the seismograms of the structure to be modeled will not be obtained.

The change in the wave velocity depending on the frequency is called Dispersion. The wave propagation becomes scattered with the increasing traveling time; this phenomenon is called Grid dispersion [4]. An adequate amount of grids must be used in the analysis of a wave length in order to avoid the grid dispersion. For this, the plane sampling number of the progressive wave front must be sufficient in terms of the wave length. The wave front progressing in the medium has the $\lambda = c/f_p$ wave length. Here, f_p is the frequency corresponding to the peak frequency of the source function. Selection of the λ/h (h: grid interval) ratio is important for eliminating the grid dispersion.

5.1. Stability Condition for the Finite Differences

The $1\text{-}2P^2$ value must be less than or equal to zero so that stable solutions can be obtained. $P \leq \pm 1/\sqrt{2}$ is found from $1\text{-}2P^2 \leq 0$. Because the P value cannot take negative values in the implementation,

$\dfrac{c\Delta t}{h} \leq \dfrac{1}{\sqrt{2}}$ must be provided in order to obtain stable solutions. A solution of the 2-D acoustic wave equation given with the formula (1) is as follows:

$$u_{i,j,k} = e^{iix} e^{ijz} e^{ikt} \qquad (40)$$

Here, x and z are real quantities, t is a complex ($t=\mu+i\delta$) quantity. If the equation (4.1) is substituted in (1) and reformulated, the equation

$$\sin^2\left(\frac{t}{2}\right) = P^2\left(\sin^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{z}{2}\right)\right) \qquad (41)$$

is obtained [22]. If a transformation is made as $\mathrm{Sin}^2\alpha=1-\cos 2\alpha/2$ and x,z,t are considered zero, $1\text{-}2P^2=0$ is obtained. Then, $P=\pm 1/\sqrt{2}$ is found. $P \leq 1/\sqrt{2}$ must be provided for stable solutions. If $P > 1/\sqrt{2}$ is considered, complex roots will be obtained and an exponentially increasing solution will be formed, where the negative imaginary roots i and j go infinite.

5.2. Stability Condition for the Fourier Transform

Some conditions must be fulfilled in order to obtain stable solutions in the Fourier method, as in the finite difference method. One of the solutions of the 2-D acoustic wave equation given in the formula (1) is

$$u = e^{ik_x x} e^{ik_z z} e^{iwt} \qquad (42)$$

If this solution is substituted in the equation (1) and reformulated,

$$c\Delta t < \frac{2}{\left(k_x^{\ 2} + k_z^{\ 2}\right)^{1/2}} \qquad (43)$$

is obtained. The biggest value that the k_x and k_z wave numbers can get is the Nyquist value and these are stated as $k_x=\pi/h$ and $k_z=\pi/h$. The stability condition is found

$$P = \frac{c\Delta t}{h} < \frac{\sqrt{2}}{\sqrt{\pi}}$$

by using these values. This given condition was obtained for the 2-D solution.

5.3. Stability Condition for the Variable Scale Method

Some conditions must be fulfilled in order to obtain stable solutions in the variable scale method, as in the finite difference method. The following condition must be provided, which is similar to the condition valid for the finite differences:

$\dfrac{c\Delta t A(\xi)}{h} \leq \dfrac{1}{\sqrt{2}}$ A(ξ) is given as the local constant here.

6. Modeling Study

Finding the effect of a structure, whose underground geometry and characteristics are known, is defined as modeling. Synthetic seismogram modeling's can be conducted with various methods. The methods mostly used in practice are based on the numeric solution of the wave equations. Because the energy propagations, which show how the wave field behaves under the ground at a specific time, can be snapshot with these methods and the source can be placed at any depth and distance. Here, the 2-D acoustic wave equation was solved with the finite differences, variable scale method and the Fourier transform, and modeling was carried out. In all the models, the Gabor wavelet with a damping constant 4 was used as the source function and the source was applied to one of the grid points when t=0.

6.1. Modeling Various Geological Structures

Solving the wave equation with numeric methods and its synthetic seismogram modeling depend on the calculation of the derivative equalities with various methods. In this study, derivative equality theories will be handled with the finite differences, variable scale method and the Fourier transforms. The points to consider in the modeling are to provide the stability condition and eliminate the effect of the grid dispersion. These effects can be kept under control with the appropriate grid and source peak frequency. It will be explained how the sampling intervals and peak frequency must be selected during the comparison of the methods. The methods used in the 2-D modeling will be compared for a simple model; the advantages and disadvantages of the methods will be presented.

6.2. Comparison of the Methods for 2-D Modeling

The usage, advantages and disadvantages of the methods will be compared in the modeling conducted through the solution of the 2-D acoustic wave equation. A two-layers medium were chosen. The velocity of the first layer was considered as 1500 *m/s*, the velocities of the second layer as 3000 *m/s* and 4200 *m/s*, time sampling interval as $\Delta t=0.25$ *ms*, plane sampling intervals as $\Delta x=\Delta z=h=10m$ for the finite differences, variable scale method and as $\Delta x=\Delta z=h=20m$ for the Fourier method (Fig. 6.2.1a and 6.2.1b). For the source term in equation (1), the first derivative of the Gaussian function is used [3, 5].

$$f(t)= (t - t_s)e^{-\alpha(t-t_s)^2} \qquad (44)$$

where the parameter α, which controls the wavelength content of the excition, is equal to 10000, and t_s is delay time parameter, is chosen as to 0.1 *s*. The source is located 100 *m* below the surface. The receivers are accommodated at 10 *m* below the surface. The grid size and time increment are 3 m and 0.001 sec respectively, which ensure accurate calculation for frequencies up to 10 Hz. As the wave must travel through sea-water before it hits any solid structures, this would be the most logical medium to use.

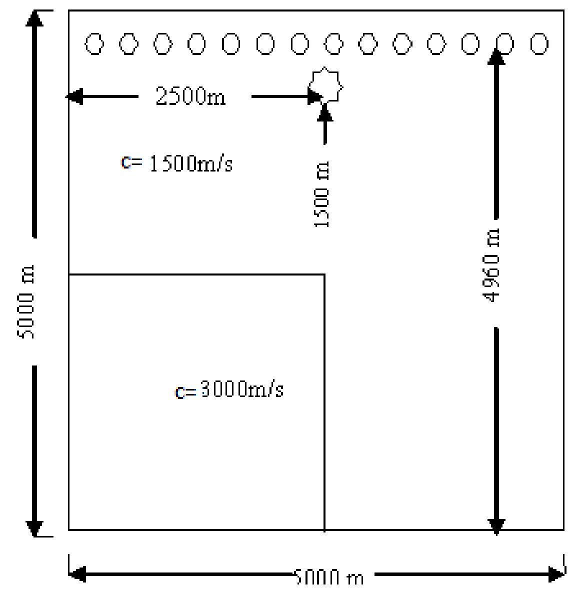

Fig.6.2.1a Physical the geometry of the corner model used to generate seismograms [23].

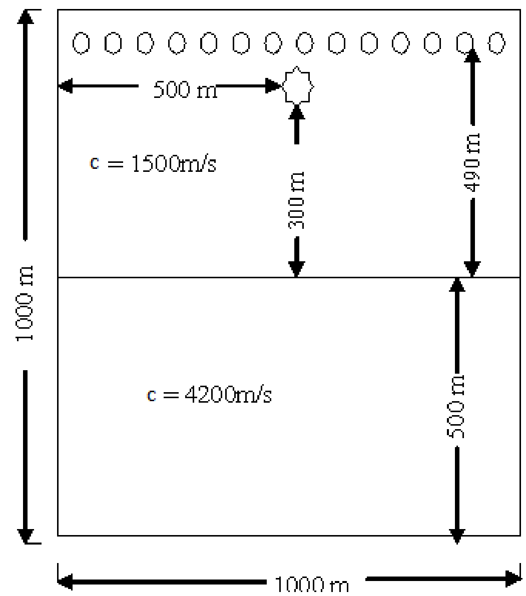

Fig.6.2.1b Physical the geometry of Two layer model used to generate seismograms [23].

The biggest velocity in the model is taken into account for finding the stability condition and the smallest velocity in the model is considered for showing the effect of the grid dispersion. Stability is $P=c\,\Delta t/h\leq 0.707$ as a condition for the finite differences and the VSM, and $p\leq 0.2$ for the Fourier transform. By considering 3000 m/s, the biggest velocity value in the selected two-layer model, p value was found for the finite differences and the variable scale method **as;**

$$P=\frac{3000.25*10^{-4}}{20}=0.375\text{ (Fig.6.2.1a)} \quad \text{and} \quad P=\frac{4200.1*10^{-4}}{10}=0.42\text{ (Fig.6.2.1b)}$$

for the Fourier method **as;**

$$P=\frac{3000.25*10^{-4}}{40}=0.1875\text{ (Fig.6.2.1a) and} \quad P=\frac{4200.1*10^{-4}}{20}=0.21\text{(Fig.6.2.1b)}$$

These values are between the stability boundaries.

The λ/h ratio must be higher than $10h$ for the finite differences and the variable scale methods to avoid the grid dispersion. But it is adequate for the Fourier method when this ratio is about 3-4 h. It is seen here that a lower number of grids will be adequate in the Fourier method compared to the finite differences. When 1500 *m/s*, the smallest velocity value in the two-layer model, is considered, a wave length of approximately 65*m* is obtained. λ/h ratio will be 13 for the finite differences, the variable scale and 6.5 for the Fourier method. The seismogram obtained with the finite differences and valuable scale method for the two-layer model is given in Fig. 6.2.2a and Fig. 6.2.3a, and the seismogram obtained with the Fourier method is shown in Fig. 6.2.2b and Fig. 6.2.3b. Because the plane sampling intervals were considered as 20*m* for the finite differences, 200 traces were obtained in Fig. 6.2.2a, and 100 traces in Fig. 6.2.2b due to the fact that the plane sampling interval was considered as 40*m* for the Fourier method. A structure, which is modeled with 200+200 grids through the finite differences, can be modeled with 100+100 or less grids through the Fourier method.

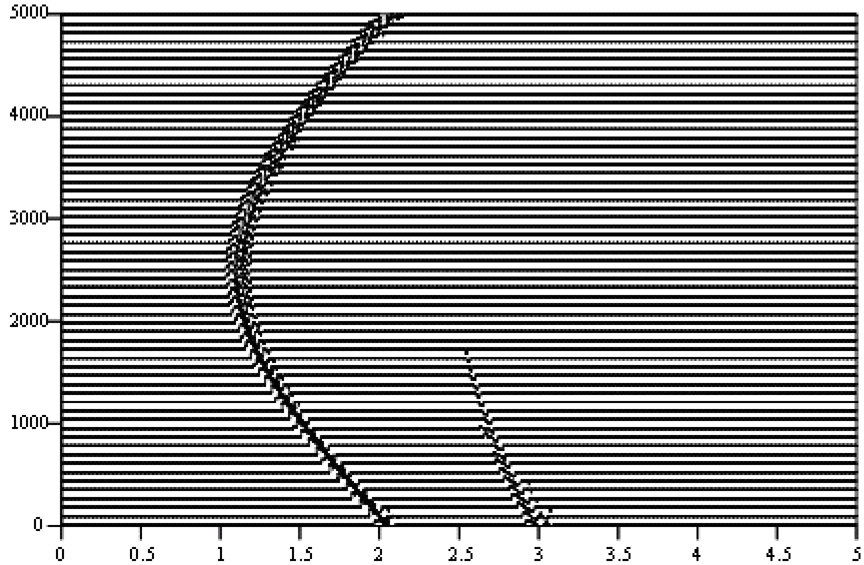

Fig.6.2.2a. Synthetic seismograms generated from the finite differences and the variable scale methods of the 2D acoustic wave equation for a medium infinite extent(Absorbing boundary conditions)

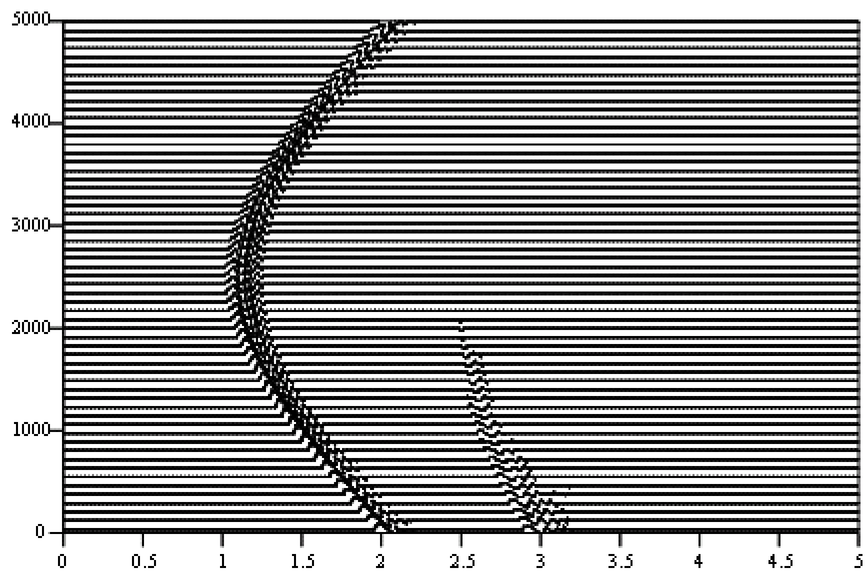

Fig.6.2.2b. Synthetic seismograms generated from the Fourier method of the 2D acoustic wave equation for a medium infinite extent(Absorbing boundary conditions)

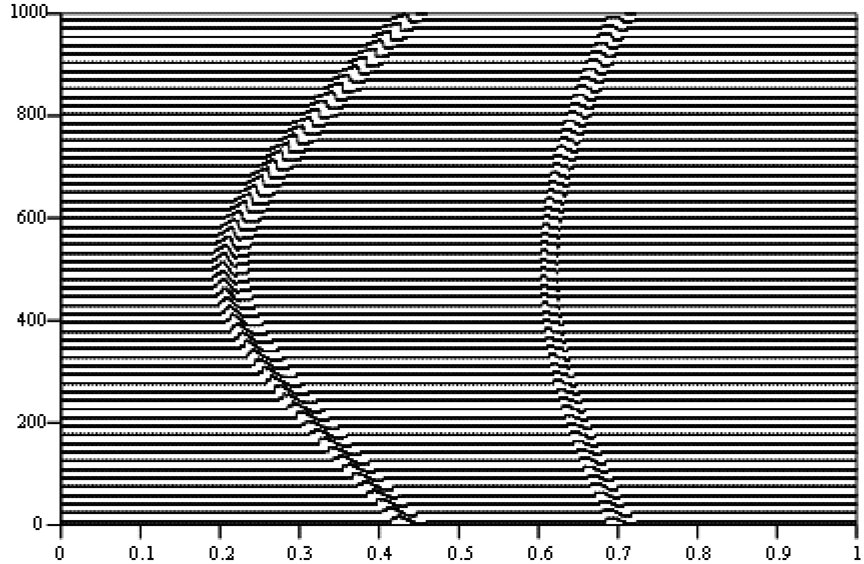

Fig.6.2.3a. Synthetic seismograms generated from the finite differences and the variable scale methods of the 2D acoustic wave equation for a medium infinite extent(Absorbing boundary conditions)

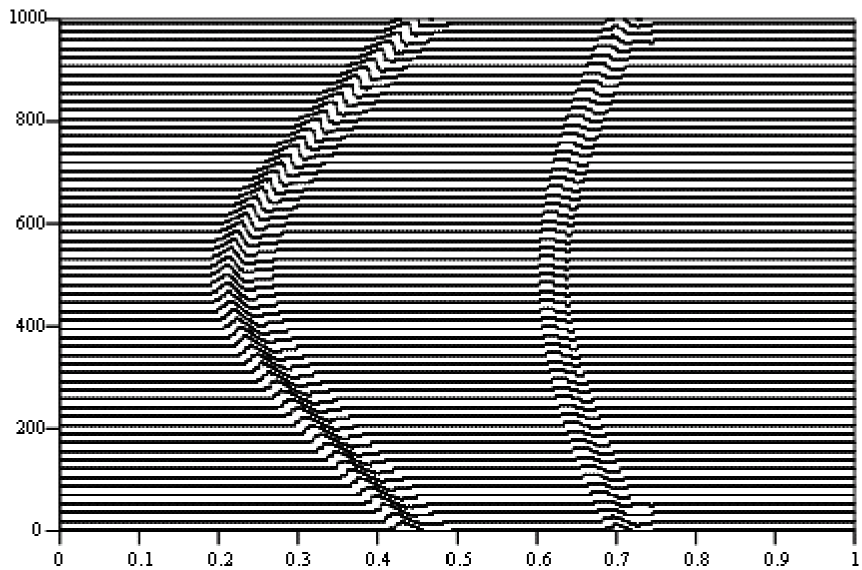

Fig.6.2.3b. Synthetic seismograms generated from the Fourier method of the 2D acoustic wave equation for a medium infinite extent(Absorbing boundary conditions)

7. Conclusions

In this study, the 2-D acoustic wave equation was solved by using the finite differences, Fourier and variable scale methods, and synthetic seismograms were modeled. It is possible in modeling's conducted with the wave equation to see how the wave field expanded at a specific time and to place the source at a desired depth and distance. Because of the fact that the Fourier method is more sensitive to the grid dispersion compared to the finite differences, 2 or 3 times more grid number is needed in modeling's conducted with the finite differences and variable scale . Phenomena are absorbed in the first and last traces due to the window boundary used in the modeling with the Fourier method. The finite differences and variable scale method will be more appropriate in modeling the complex structures, because phenomena can be observed in all the traces within the finite differences and variable scale method.

In the modeling's conducted with the acoustic wave equation valid for a homogenous and isotropic medium, the decrease seen in the energy can be observed in seismograms due to the first arrivals, reflections, repeatable reflections, diffractions, global expansion and reflection coefficients. When the modeling of the mode transformations and absorption is intended, the wave equations that are appropriate for the heterogeneous medium must be analyzed.

REFERENCES

[1] Alterman Z. S. Propagation Elastic Wave in Layered MEdia by Finite Difference Methods, Bull. SSS, 58, (1968), 367-389

[2] Ottaviani M., Elastic Wave in Two-Evently- Welded Quarter Apaces, Bull. SSA, 61, (1971) , 1199-1211

[3] Alford R. M., Kelly K.R. and Boore D.M., Accuracy of Finite Difference Modeling of The Acoustic Wave Equation, Geophysics, 41, (1976), 834-842

[4] Kelly K.R., Ward R.W., Treitel S. And Alford R.M., Synthetic Seismograms; A Finite Difference Aproach, Geophysics, 41, (1976), 2132-2157

[5] Reynolds A.C., Boundary Condition for Numerical Solution of Wave Propagation Problems, Geophysics, , 43, (1978), 1099-1110

[6] Smith W.D., The Application of Finite-Element Analysis to Body Wave Propagation Problems, Geophys. Jour. Roy. Astro. Soc., 42, (1975), 747-755

[7] Gazdag J., Modeling of The Aoustic Wave Equation with Transform Methods, Geophysics, 46, (1981), 854-859

[8] Fornberg B., The Pseudo-Spectral Methods; Comparisons with Finite Differences for Elastic Wave Equation, Geophysics, 47, (1987), 483-501

[9] Kossloff D., and Baysal E., Forward Modeling by A Fourier Method, Geophysics, 47, (1982), 1402-1412

[10] Kossloff D., REshef M., and Lowenthal D., Elastic Wave Calculation by The Fouier Methods, Bull. SSA., 74, (1984), 875-888

[11] Saatçılar R., Ergintav S. and Canıtez N., The Use of Hartley Transform In Geophysical Applications, Geophysics, 56, (1991), 1488-1495

[12] Saatçılar R. and Ergintav S., Solving Elastic Wave Equations with Hartley Method, Geophysics, 56, (1991), 274-278

[13] Demir, İ., Seismic Wave Modelling Using Finite Difference Methods, Ph.D. Thesis, University of Glamorgan, UK., (1998)

[14] Jensen P. S., Finite Difference Technique for Variable grids, Computer&Structures, 2, (1972) 17-29

[15] Perrone N. and Kao R., A General Finite Difference method for arbitrary meshes, Computer&Structures, 5, (1975) 45-58

[16] Demir, İ. 2002. Variable-Scale Methods For Seismic Wave Propagation in homogeneous media, Bulletin of Pure and Applied Sciences, Vol. 21-E, No:1, pp.221-226.

[17] Benito J. J., Urena F., Gavete L. Salete E. and Muelas A., A GFDM with PML for Seismic Wave Equations in Heterogeneous Media, Journal of Computational and Applied Mathematics, 252, (2013), 40-51

[18] Baysal E., Modeling and Migration by The Fourier Method, Ph.D. Thesis, Houston University, Houston, (1982)

[19] Smith G.D. Numerical Solution of Partial Differantial Equations with Exercises ans Worked Solution, Second Edistion, Oxford University Pres, Oxford, (1982)

[20] Clayton R.W. and Enquist B., Absorbing Boundary Conditions for Acoustic and Elastic Wave Equations, Bull. SSA., 67, (1977) 1529-1539

[21] Cerjan C., Kossloff D., Kossloff R. And Reshef M., A Nonreflecting Boundary Condition for Discreate Acoustic and ELastic Wave Equations, Geophysics, 50, (1985), 705-709

[22] Mitchell A.R., Comppututional Methods in Partial Differantial Equations, Aberden Univ. Press, Aberden (1969).

[23] Hall F. and Wang Y., Elastic Wave Modelling by an Integrated Finite Difference Method, Geophysics Journal Int., 177, (2009) 104-114.