Logistic Differential Equations and Chaos Theory

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Abstract

Nowadays, we can see chaotic behaviors in many areas. One day a slight change in atmospheric conditions can lead to completely different weather conditions after a few days. In most of these air systems, a butterfly flapping its wings on one continent can cause hurricane on another continent. Earthquake, weather forecasting, physics, astronomy, medicine, biology, chemistry, informatics and news technology, as well as sociology, ecology and economics are among the main areas where such chaotic situations occur. Minor differences in initial conditions (such as rounding errors in numerical calculation) give broad and different results for such dynamic systems. This may make the long-term estimation of their behavior impossible. Although these systems are deterministic, that is, although they are determined according to the initial conditions of future behavior, they do not contain random elements. In other words, the deterministic nature of these systems does not make them predictable. This behavior is simply known as chaos. When we encounter such situations, we would like to model and predict the results of such situations according to our needs.

In this study, we examined various logistic differential equations studied in order to model such situations. We compared exponential and logistic differential models with each other and examined their advantages and disadvantages. Finally, we examined the interaction of two species in the same environment with Lotka-Volterra equation. We tried to model the problems we face today through logistic differential equations.

Keywords: Logistic Growth Equations, Logistic Differential Equations, Exponential Models, Chaos Theory, Lotka-Volterra Differential Equations.

1. Introduction

The first steps in the theory of chaos were taken in the 1960s when a scientist named Edward Lorenz created a weather forecast model on his computer at the MIT (Massachusetts Institute of Technology). This model consisted of a complex set of formulas. Meanwhile, Lorenz's students and colleagues were observing this model thoroughly and were amazed by it. Because this algorithm, Lorenz's prediction model, did not seem to repeat an array or data. On the contrary, the actual weather conditions varied. Because of the 'Uncertainty Principle' there was no such thing as knowing the changes in these weather conditions and the behavior of the air. We could only get an approximate value with the data we had, and sometimes these data would not be useful for the future. Because small changes in weather conditions would have faded the forecasts. Thus, Chaos Theory emerged. Then Robert May argued in 1975 that population growth could cause chaos[2].

Mathematicians such as Vladimir Arnold, Stephen Smale and Robert May suggested that this chaos can be expressed by differential equations and began to work on logistic differential equations. In this study, we examined the types of logistic differential equations and the examples of these equations. We examined the difference between the Exponential Model and Logistic Model which are one of the population growth models. We then worked on the growth of multiple species and especially the predator model of Lotka-Volterra and examined the effects of multiple species on each other's survival and growth[3].

2. Materials and Methods

First, the growth of a single species was discussed. Afterwards, the growth of different species in the same environment, their interactions with each other and the population growth of predator species were examined. In exponential growth of a single species, the exponential and logistic models were compared. Lotka-Volterra model was used to examine the growth of multiple species.

2.1. Logistic Differential Equations

Unlike the exponential growth model, the general formula of logistic differential equations with K carrying capacity

$$\boldsymbol{P'(t) = r\,P(t)\left[1 - \frac{P(t)}{K}\right]} \qquad \textbf{(2.1.1)}$$

2.2.Population Growth of a Single Species

Let P (t) symbolize the amount of a population at any time t. For example, specify the number of human population in the world. The simplest population growth model is the Exponential model. In this model, let's use the constant r as the growth parameter.

$$\boldsymbol{P'(t) = rP(t)} \qquad \textbf{(2.2.1)}$$

is an exponential differential equation. Because if we take the integral of the differential equation, then we will get

$$P(t) = P_0 \, e^{rt}$$
$$P_0 = P(0) \qquad \textbf{(2.2.2)}$$

initial population.

But the important feature of the exponential model is that the population increases continuously when r is positive.

Such growth is acceptable in the first place, but in later times the exponential growth model appears to be inadequate and inaccurate. Because the population in a country or the population of bacteria in a container cannot grow exponentially without fluctuation. Verhulst, who saw this insufficiency, proposed an alternative model to the exponential model in 1836.

This proposal laid the foundations of the current Logistic differential equation. According to this model, the equation contains 2 parameters. The first parameter is the r parameter, which is also called the growth parameter, and the second parameter is the load capacity parameter called K. Solution of logistic differential equation

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}$$

$$P_0 = P(0) \qquad \textbf{(2.2.3)}$$

initial population. In equation (2.2.3) t goes to infinity, we can see easily e^{-rt}goes to 0. Hence we got

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}$$
$$\lim_{t \to \infty} P(t) = \frac{P_0 \, K}{P_0} = K$$

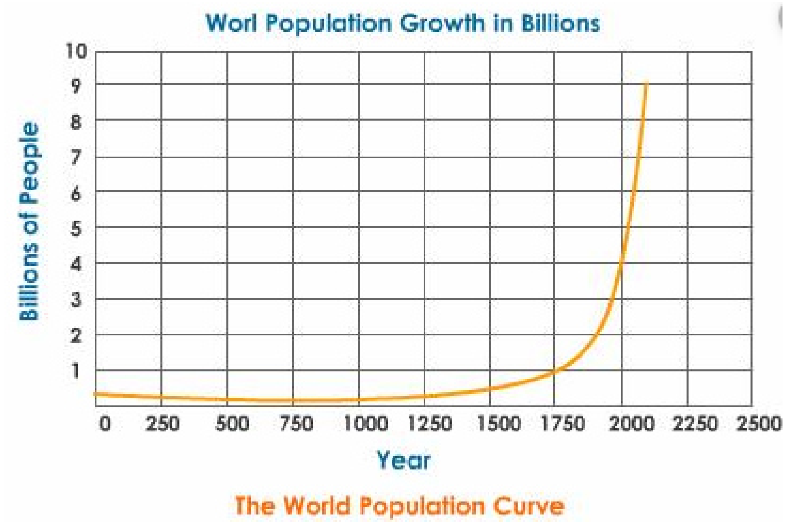

Figure 2.1. Estimated world population between 0 -2500

As seen in R. Saygılı's 2014 survey, the world population does not show full exponential growth (Figure 2.1.). Therefore, it is exemplified by the graph above that growth modeling of a single population cannot be fully explained by an exponential model.

2.3. Population Growth of a Multiple Species

One of the most well-known population growth models of multiple species is the Lotka-Volterra equations, which are called the Prey-Predator model. This model is a nonlinear differential equation which is commonly used to describe the dynamics of biological systems interacting with one predator and the other with prey[4].

Assuming that the prey has an unlimited number of foods and never meets the prey, we can imagine that the prey population has multiplied exponentially. This increase can be represented by the term αx in the exponential growth equation. The hunting rate in prey is represented by the term βxy, which is related to how much the prey and the hunter confronts. The variation in the number of prey populations is found by subtracting the prey from the growth rate of the prey[5].

If we look at the predator, let δxy represent the growth of the predatory population. The absence of the predator (such as natural death, migration) is symbolized by γx. By looking at these parameters, the rate of change of the hunter population is found by subtracting the natural death and migration parameters from the growth rate of the hunter.

Suppose that there are two different kinds of animals, one wolf (hunter) and the other rabbit (prey). Suppose there are 10 wolves and 10 rabbits as a starting condition. Let the growth and mortality rates of rabbits be 1.1 and 0.4. The growth rate of wolves is 0.1 and the mortality rate is 0.4. Let's draw the variation of the population number with these random parameters.

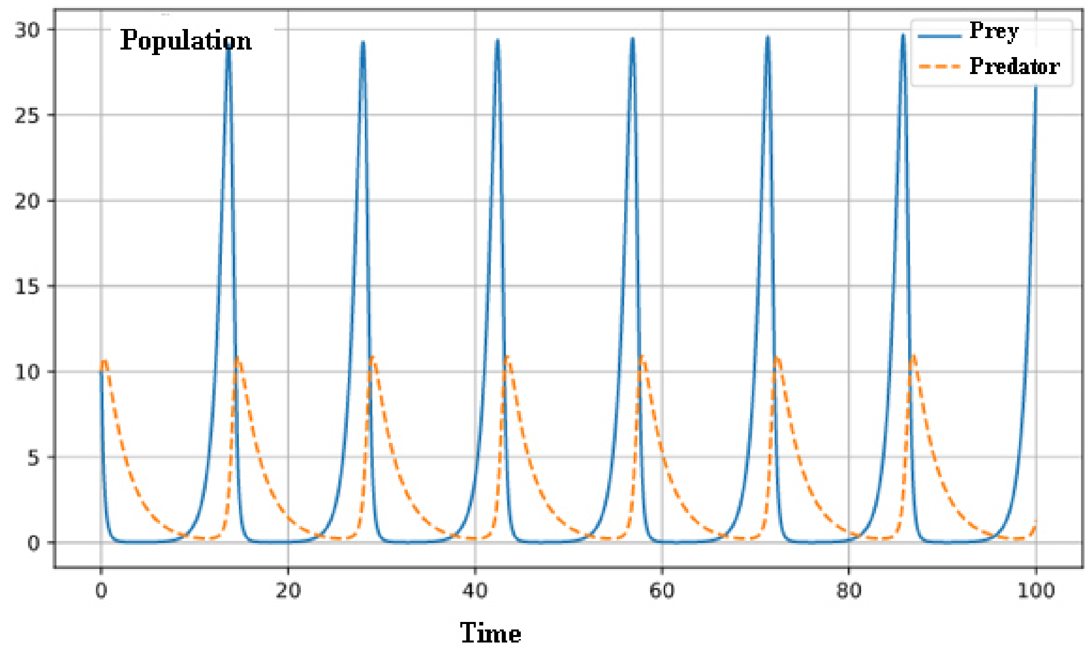

Figure 2.2. Population graph of 10 wolves and 10 rabbits in the same environment

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - y}{\beta y - \alpha} \qquad (2.3.1)$$

[1]

If we distribute the y parameter over δx and γ in the equation, we get the term δxy - x, which gives us the rate of change in the predator population. In the denominator part, if we distribute the $-x$ parameter over the term $\beta y - \alpha$, this gives us the rate of change of the prey population. Then (2.2.1) If we take the integral of the equation we get;

On the other hand we got

$$V = \beta x - \gamma ln(x) + \beta y - \alpha ln(y) \qquad (2.3.2)$$

[1]

Theorem: $P_0 = P(0)$solution of initial differential equation is

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}} \qquad (2.3.3)$$

Example : Let's assume that the World population has increased according to the logistic differantial equation. If $K = 15$ billion people, $r = 0{,}013$ and $P(0) = 6{,}122$ billion people. How much will the World population be in 2100?

Solution : The value of the world population in 2000 is used for the starting population. The question is the world population in 2100. Therefore, the value of the parameter we set as time is t = 100 years.

$$\boldsymbol{P(t) = \frac{P_0\,K}{P_0 + (K - P_0)e^{-rt}}}$$

$$\boldsymbol{P(100) = \frac{6,122.15}{6,122 + (15 - 6,122)e^{-0,013.100}}}$$

$$\boldsymbol{P(100) = \frac{91,83}{6,122 + 8,878.e^{-1,3}}}$$

$$\boldsymbol{P(100) = \frac{91,83}{6,122 + 2,433}}$$

$$\boldsymbol{P(100) = \frac{91,83}{8,555}}$$

$\cong \mathbf{10.7}$ *billion*

3. Conclusion

Logistic differential equation model gives more realistic data with K carrying capacity compared to the exponential model in population growth modeling of a single species. As seen in Figure 2.1, the world population is approaching the capacity of carrying K in the formula of logistic differential equation rather than increasing exponentially. In the growth of multiple species, Lotka-Volterra equation, prey-predator modeling shows a better approach. Considering the world population, the world's carrying capacity is estimated to be around 8-16 billion according to most scientists, with the shortage of food and the lack of freshwater resources.

According to the report titled "World Population Expectation" prepared by the United Nations (UN), the world population, which is currently around 7.7 billion, will increase by two billion in 2050 to 9.7 billion. The experts who prepared the report also estimate that the world population will be 11 billion by 2100 if there are no extraordinary events and world population control cannot be achieved. Of course, as a result of the decrease in the amount of nutrients in the past years and the rate of population growth not constantly constant and decreasing and progressing, the result that will be obtained in this thesis will be different with the amount we predicted for 2100 in this thesis. For this reason, we cannot look at the results we have found regarding the population amount as definitive results. It is understood that logistic differential equations are one of the most important equations that will be used to cope with and anticipate the problems of population growth, which is likely to come up in the coming years.

It was emphasized that using logistic equations would be a great advantage for us to have a prediction about the transmission of certain diseases and epidemics.

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